- Scale analysis
- Angular momentum and inertial stability
- Tropical cyclones as Carnot heat engines
- Potential intensity

Goal: Develop simple dynamic and thermodynamic arguments to explain key features of tropical cyclones

Tropical cyclone scale analysis (I)

The momentum equations in cylindrical coordinates (r,ϕ,z) are:

Radial:

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} + w\frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r} - fu_\phi = -\frac{1}{\rho_o}\frac{\partial p}{\partial r}$$

Azimuthal:

$$\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + w\frac{\partial u_\phi}{\partial z} + \frac{u_\phi u_r}{r} + fu_r = -\frac{1}{\rho_o r}\frac{\partial p}{\partial \phi}$$

Vertical:

$$\frac{\partial w}{\partial t} + u_r \frac{\partial w}{\partial r} + \frac{u_\phi}{r}\frac{\partial w}{\partial \phi} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_o}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_o}g$$

Let’s now consider a scale analysis appropriate to tropical cyclones under the following assumptions: (i) the flow is steady and frictionless; (ii) the flow is axisymmetric; (iii) the vertical motion scale is small compared to horizontal motion (W < U) and the flow is hydrostatic. For the azimuthal equation under these assumptions:

$$u_r\left(\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r} + f\right) \approx 0 \Rightarrow u_r = 0$$

Tropical cyclone scale analysis (II)

Let's now consider the radial equation under these assumptions:

$$\frac{u_\phi^2}{r} + f u_\phi = \frac{1}{\rho_0}\frac{\partial p}{\partial r}$$

As in prior scale analysis, we'll consider the above equation rescaled by the inertial scale U^2/L

$$\frac{u_\phi^2}{r} + f u_\phi = \frac{1}{\rho_0}\frac{\partial p}{\partial r}$$

$$1 \qquad \mathrm{Ro}^{-1} \qquad \frac{\Delta P}{\rho_0 U^2}$$

For three regions of interest within the cyclone—eye, area enclosing gale force winds, and outer edges—the typical length and velocity scales are:

Region	Length Scale (m)	Velocity Scale (ms^{-1})
Eye	10^4	10
Gale force region	10^5	10
Outer edges	10^6	1

Tropical cyclone scale analysis (III)

Assuming a latitude of 15 degrees gives f~10^{-5}. Then

Region	**Ro^{-1}**	**Balance**
Eye	10^{-2}	Cyclostrophic
Gale force region	10^{-1}	Gradient
Outer edges	10	Geostrophic

These results indicate that the dynamical balance in the radial equation differs depending on the region of the cyclone under consideration. Note that the simplified radial momentum equation can be solved $u\phi$ via the quadratic equation:

$$u_\phi = \frac{fr}{2}\left[-1 \pm \sqrt{1 + \frac{4(\partial p / \partial r)}{\rho_0 f^2 r}}\right]$$

Observed azimuthal wind versus gradient wind estimate

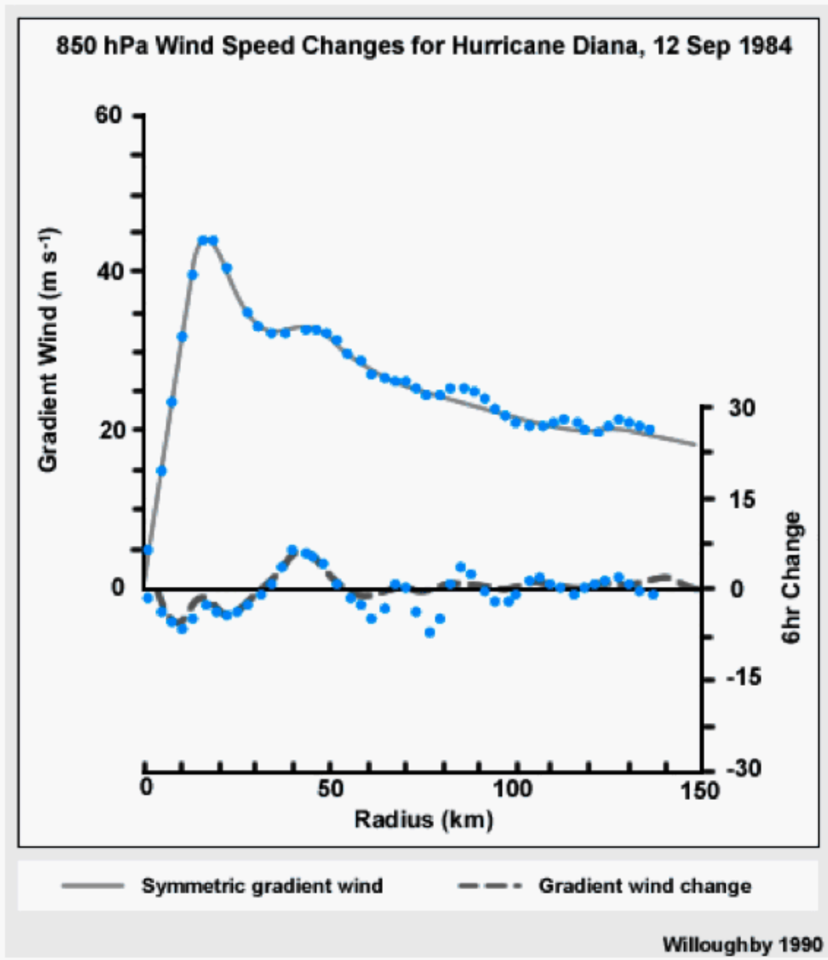


Fig. 10.8. 850 hPa wind observations (dots) for Hurricane Diana on 12 Sep 1984, symmetric gradient wind (solid) and gradient wind change (dash) deduced from the observed height field. Adapted from Willoughby (1990).[22]

Tropical cyclone absolute angular momentum

Recalling the definition of angular momentum:

$$\vec{M} = \vec{r} \times \vec{v}$$

For the radius equal to the cyclone radius, the angular momentum about the local vertical at the center of the cyclone consists of two contributions, the cyclonic wind field plus the contribution from the Earth's rotation. Thus:

$$M_{z'} = \left(\vec{r} \times \vec{v}\right)_{z'}$$

$$= \left[R\hat{r} \times \left(v\hat{\phi} + \Omega\hat{z} \times R\hat{r}\right)\right]_{z'} = Rv + \left[R\hat{r} \times \left(\Omega \sin\lambda \hat{z}' \times R\hat{r}\right)\right]_{z'}$$

$$= Rv + \Omega \sin\lambda R^2 = Rv + \frac{fR^2}{2}$$

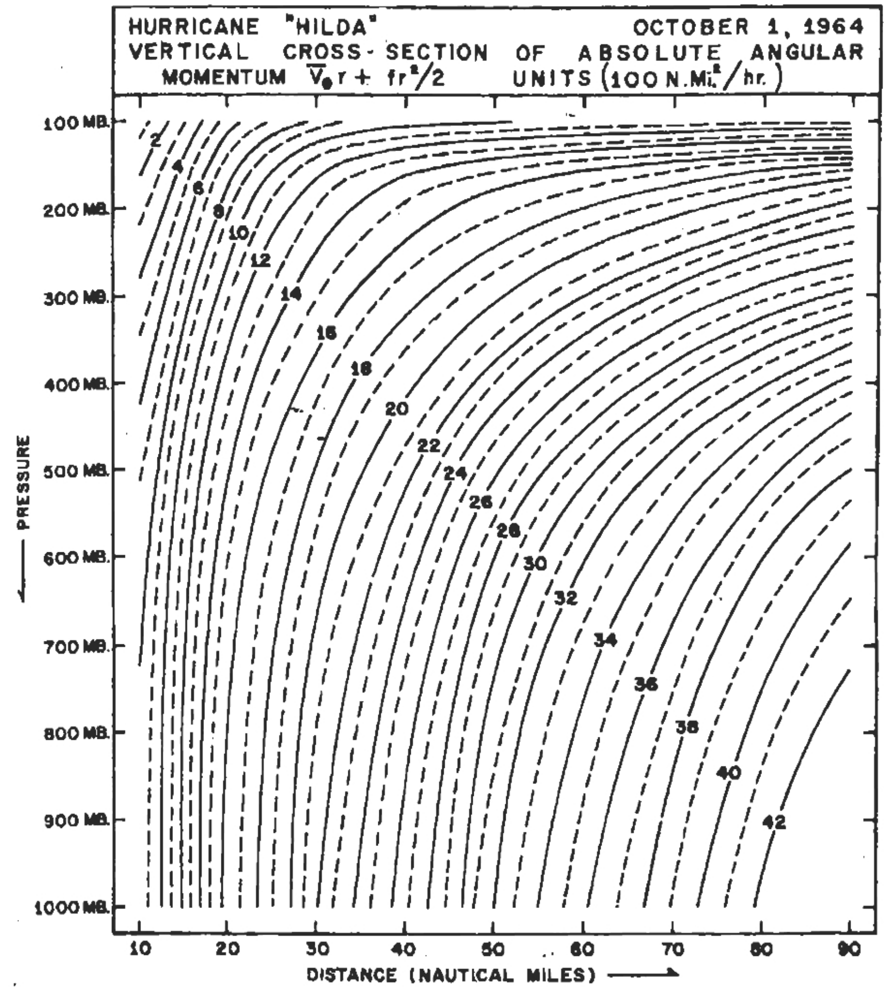
Inferred structure of angular momentum

For *M(r,z)*, we note the condition for constant *M*:

$$dM = 0 = \left.\frac{\partial M}{\partial r}\right|_z dr + \left.\frac{\partial M}{\partial z}\right|_r dz$$

$$\Rightarrow \left.\frac{dz}{dr}\right|_M = -\frac{\left.\partial M/\partial r\right|_z}{\left.\partial M/\partial z\right|_r}$$

The azimuthal wind decreases upwards, so $\partial M/\partial z < 0$. The contribution to *M* from the Earth's rotation gives *M* increasing with *r* at fixed *z*. On the other hand, the azimuthal wind increases from the center of the eye to a maximum value at the eyewall and then decreases thereafter. At small radii, then, $\partial(rv)/\partial r > 0$, while at large radii, $\partial(rv)/\partial r < 0$ is typically small. Putting this together yields constant *M* surfaces that slope upwards and outwards.



Inertial stability

The inertial stability, *I*, provides a measure of the "resistance" of a tropical cyclone (or any vortex) to changes from different forcings:

$$I^2 = (\zeta + f)\frac{2M}{R^2} = (\zeta + f)\left(f + \frac{v}{2R}\right)$$

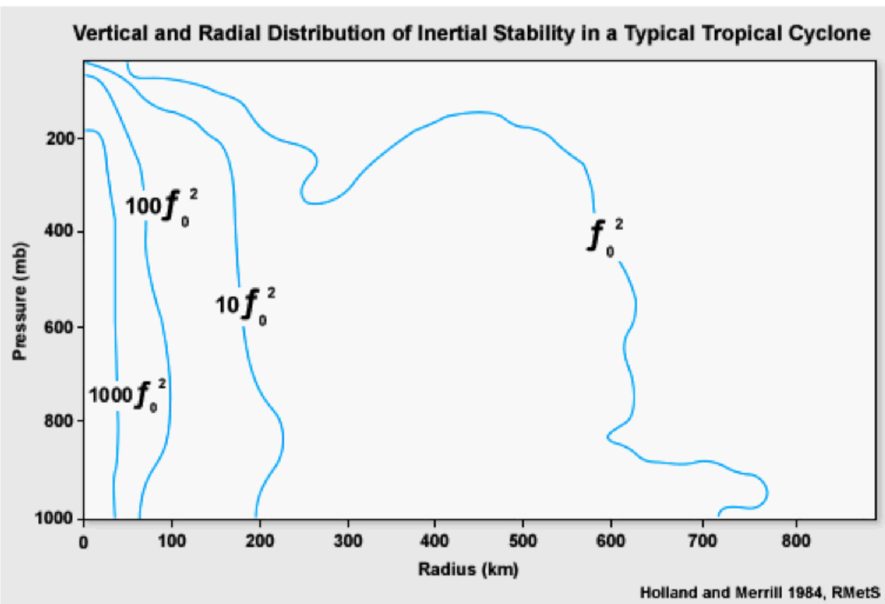
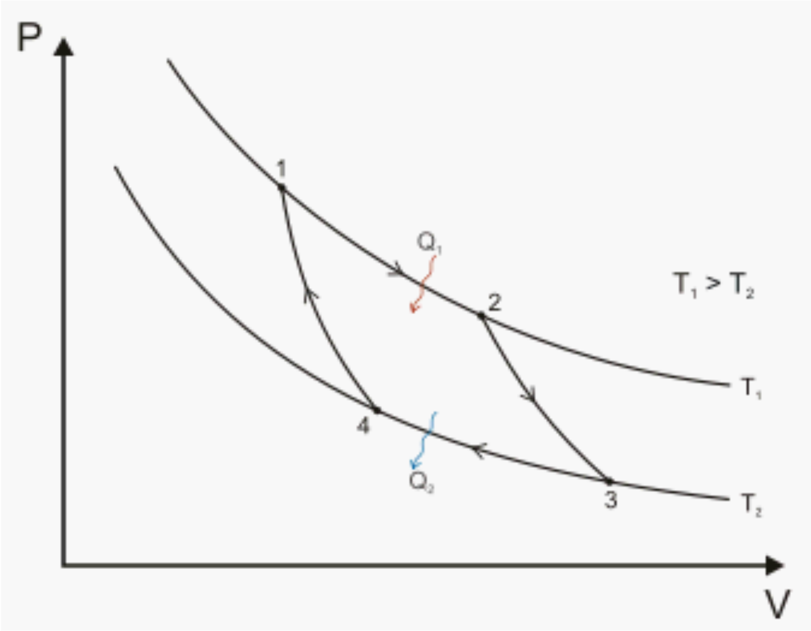


Fig. 10.10. Vertical and radial distribution of inertial stability in a typical tropical cyclone. Following Holland and Merrill (1984).[27]

For a contracting and/or intensifying storm (at a fixed latitude), I^2 increases, indicating increased inertial stability, i.e., the storm is more resistant to change. For low levels within the intensifying tropical cyclone, the relative and planetary voriticies are in the same direction, thus stabilizing the cyclone to lateral perturbations. At the top of the tropical cyclone, the flow is anticyclonic, so v (or ζ) and f are in opposite directions; thus, the intensifying cyclone's increasing inertial stability doesn't extend all the way to the tropopause.

Carnot cycle

Consider the PV diagram on the right consisting of two isotherms T_1 and T_2 (curves 1-2 and 3-4; $T_1 > T_2$) and two adiabats (curves 2-3 and 4-1). Recall the differential 1st Law of Thermodynamics:



$$dU = dQ + dW = TdS - PdV$$

Let's evaluate the 1st Law for the entire cycle.

For the isothermal curves 1-2 and 3-4, $dU=0$, so $W_1 = Q_1$ and $W_3 = Q_3$. For the adiabatic curves 2-3 and 4-1, $dQ = 0$, so $W_3 = U(T_2)-U(T_1)$ and $W4=U(T_1)-U(T_2)$. The efficiency η of a thermodynamic cycle is defined as:

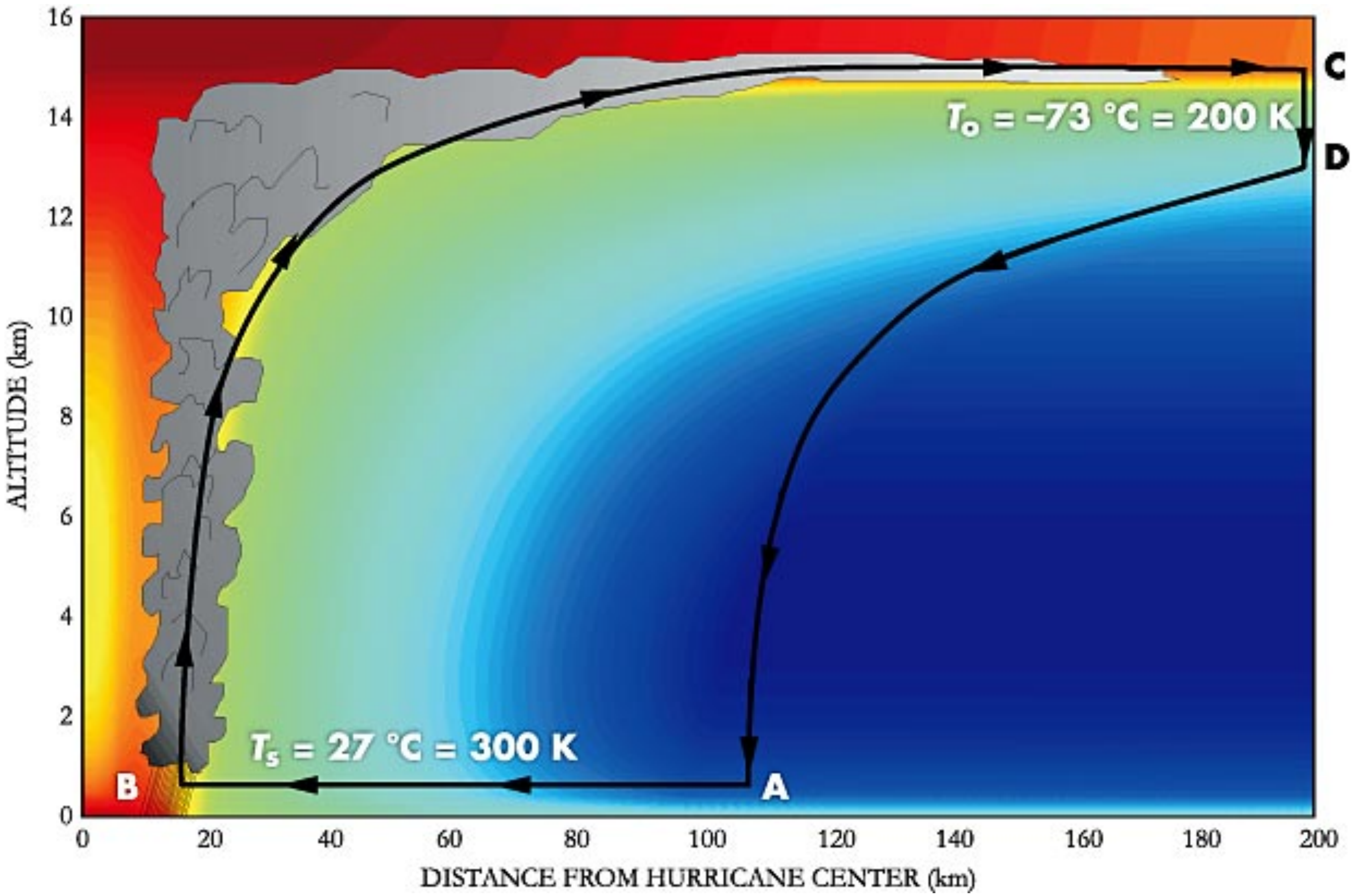
$$\eta = \frac{W}{Q_{absorbed}}$$

Thus, for the Carnot cycle:

$\Delta S_1 = -\Delta S_2$ since 2-3 and 4-1 are adiabats.

$$\eta = \frac{\sum_{i=1}^{4} W_i}{Q_1} = \frac{Q_1 + Q_3}{Q_1} = \frac{T_1\Delta S_1 + T_2\Delta S_2}{T_1\Delta S_1} = 1 - \frac{T_2}{T_1}$$

Tropical cyclone as a Carnot cycle (I)



Note: In an adiabatic system, fluid flows along surfaces of constant angular momentum

Emanuel 1996

Tropical cyclone as a Carnot cycle (II)

Assuming the Carnot efficiency, the rate of work done by the cyclone is:

$$\dot{W} = \left(1 - \frac{T_o}{T_s}\right)\dot{Q}_{absorbed}$$

The rate of absorbed heating of the hurricane is equal to the surface enthalpy flux (mostly latent heating) and frictional dissipation:

$$\dot{Q}_{absorbed} = 2\pi \int_0^R \rho\left[C_k \,|\,\mathrm{v}\,|\left(k_0^* - k\right) + C_D \,|\,\mathrm{v}\,|^3\right] r dr$$

Under steady state conditions, the work done balances frictional dissipation:

$$\dot{W} = 2\pi \int_0^R \rho C_D \,|\,\mathrm{v}\,|^3 \; r dr$$

For a profile of wind strongly peaked near the eyewall, the integrals can be approximated using values of the maximum wind, so:

$$2\pi\rho C_D \left|\mathrm{v}_{\max}\right|^3 \approx \left(1 - \frac{T_o}{T_s}\right) 2\pi\rho\left[C_k \left|\mathrm{v}_{\max}\right| (k_0^* - k) + C_D \left|\mathrm{v}_{\max}\right|^3\right] \Rightarrow \left|\mathrm{v}_{\max}\right|^2 \approx \frac{C_k}{C_D}\frac{T_s - T_0}{T_s}(k_0^* - k)$$

More detailed treatment

Using conserved variables (energy, moist entropy, and angular momentum) above the ABL, where the flow is axisymmetric, reversible, and adiabatic, and applying principles of convective quasi-equilibrium, it can be shown that:

$$V_{\max}^2 \approx \frac{C_k}{C_D}\left[\frac{\frac{T_s - T_0}{T_0}(h_s^* - h_a^*) - \frac{T_s}{4T_0} f^2 r_{outer}^2}{1 - \frac{C_k}{C_D}\left(\frac{T_s}{2T_0} - b\right)}\right]$$

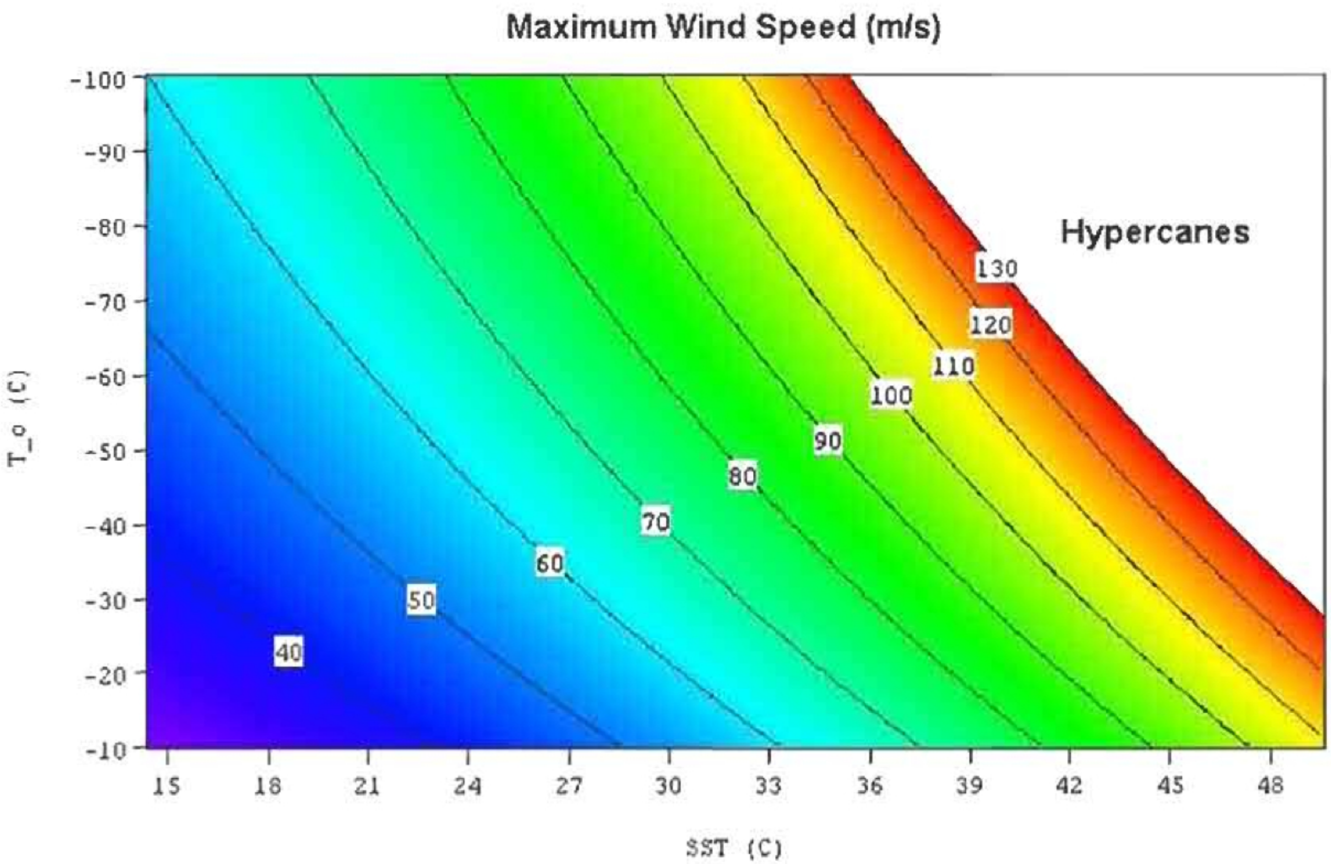
Here, r_{outer} is the outer radius of the storm, where the azimuthal velocity is 0 and pressure is equal to the ambient environment. *b* is an empirical constant relating maximum wind and pressure.

From the above expression, we can estimate a maximum upper bound on storm size, i.e., setting V_{max} to zero gives:

$$r_{outer\,\max} = \sqrt{\frac{4\left(T_s - T_0\right)\left(h_s^* - h_a^*\right)}{f^2 T_s}}$$

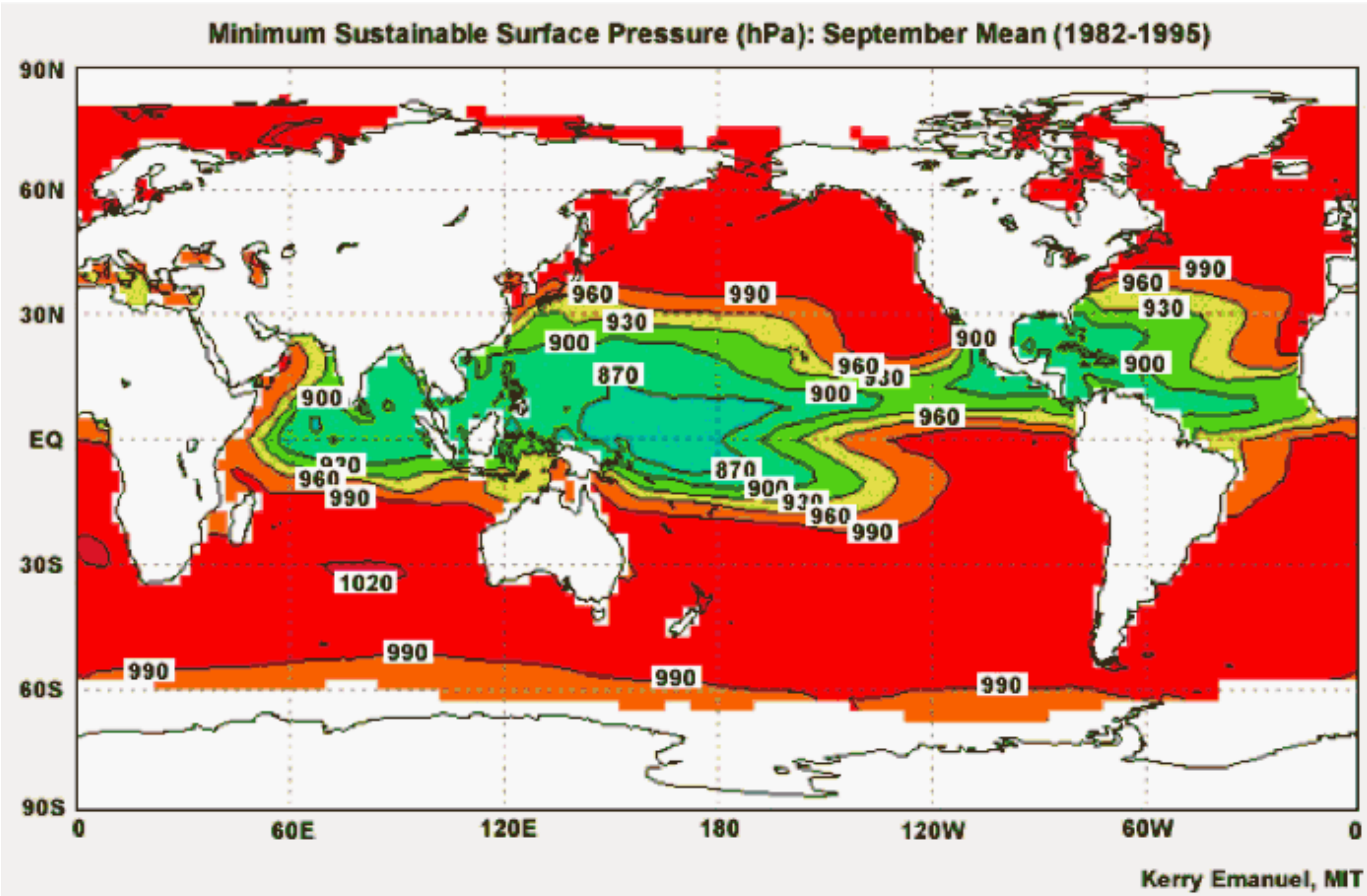
Potential intensity

The framework sketched in the previous slide can be used to estimate the potential intensity, the cyclone maximum windspeed/minimum central pressure that may be anticipated from environmental conditions.



$\mathcal{H}$ = 0.75 C_k/C_D = 1.2

Minimum sustainable sea level pressure



Comparison of observed and potential intensities

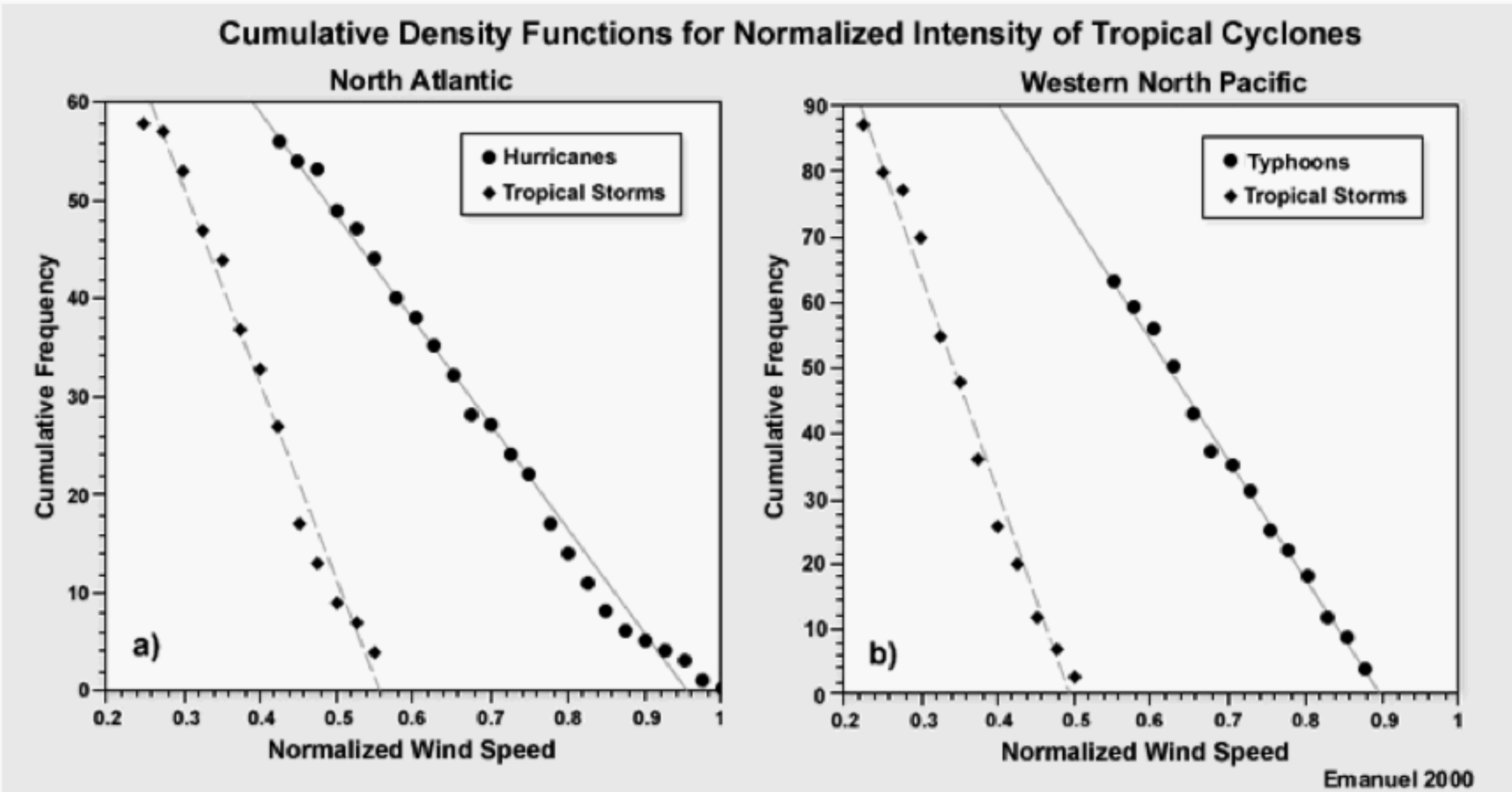
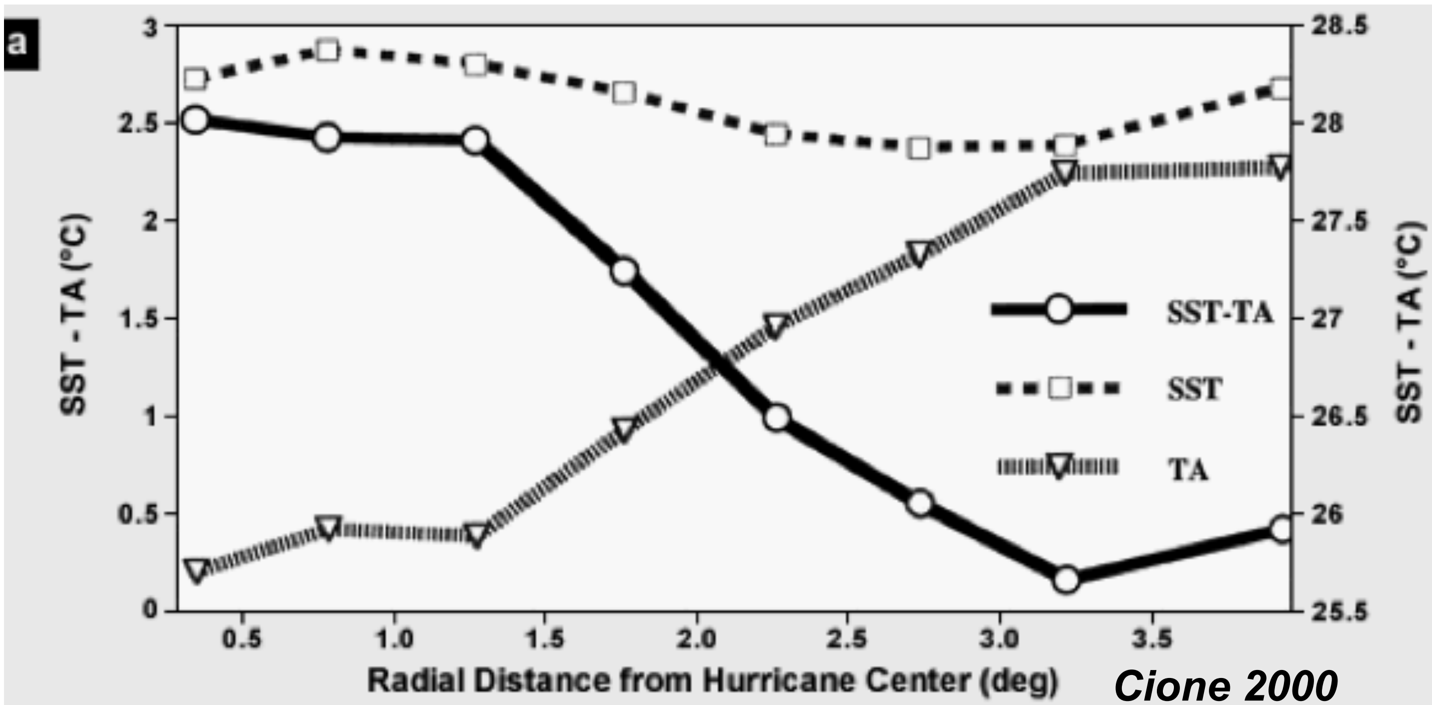


Fig. 10.45. Cumulative density functions (CDFs) for normalized intensity (actual/PI) for tropical cyclones in the North Atlantic and western North Pacific. Data are stratified by storm intensity. Only storms whose PI is not decreasing are included.

Challenges to Carnot theory



For example: the tropical cyclone boundary layer is only approximately isothermal in the high wind/strong pressure gradient region near the eye of the storm.